International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Extension Properties of States on Operator Algebras

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Received March 28, 1995

We summarize and deepen some recent results concerning the extension problem for states on operator algebras and general quantum logics. In particular, we establish equivalence between the Gleason extension property, the Hahn–Banach extension property, and the universal state extension property of projection logics. Extensions of Jauch–Piron states are investigated.

1. INTRODUCTION AND BASIC NOTIONS

The aim of this paper is to exhibit some recent results concerning states on operator algebras and their projection logics. Generally speaking, extension theorems play an important role in many fields of mathematics. As examples one can take the Hahn–Banach theorem on the extension of linear functionals and Gleason-type theorems allowing one to extend a probability measure from projection logic to a given operator algebra. The former is a basic tool for functional analysis, the latter is a central principle for the noncommutative measure theory based on operator algebras. Studying the extension problem between projection logics of operator algebras and general quantum logics, we establish the equivalence between these basic principles. This equivalence provides new insight into the Gleason theorem (Bunce and Wright, 1984, 1989) and may contribute to the noncommutative measure theory. Moreover, we show that many operator algebras (e.g., von Neumann algebras) enjoy the universal state extension property. These results strengthen both the generalized Gleason theorem (Bunce and Wright, 1984, 1989; Gleason, 1957; Christensen, 1982; Yeadon, 1984) and extension results in Pták (1985).

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In the final part of this note we shall consider the problem of extending Jauch–Piron states. Results of this section generalize results in Amman (1989) and Raggio (1981).

Let us first recall a few notions and fix the notation. Throughout the paper let $(M, \circ)$ be a *JB-algebra*, i.e., a real Banach algebra M with Jordan product $\circ$, whose norm satisfies the following conditions:

1. $\|a^2\| = \|a\|^2$.
2. $\|a^2\| \leq \|a^2 + b^2\|$ for all $a, b \in M$.

We will assume in the sequel that M is *unital*, i.e., possessing a unit element 1. [For the theory of JB-algebras we refer to the monograph Hanche-Olsen and Stormer (1984)]. Well-known examples of JB-algebras are self-adjoint parts of C^*-algebras equipped with a Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. If M is simultaneously a dual Banach space, then M is called a *JBW-algebra*. This category comprises, e.g., self-adjoint parts of von Neumann algebras. A *JW-algebra* is a weakly closed Jordan subalgebra of the self-adjoint part of a von Neumann algebra. By $P(M)$ we shall denote the set of all idempotents of M. (Let us recall that $p \in M$ is an idempotent if $p^2 = p$). Endowed with the ordering $p \leq q$ if and only if $p \circ q = p$ and the operation $p^\perp = 1 - p$, we get an order structure $(P(M), 0, 1, \leq, \perp)$ (called *projection logic*), whose properties are generalized in the following notion.

The (*quantum*) *logic* is a partially ordered set $(L, 0, 1, \leq, \perp)$ with an orthocomplementation relation $\perp$ satisfying the following conditions:

1. L has a least and a greatest element 0 and 1, respectively.
2. $a \leq b$ implies $b^\perp \leq a^\perp$.
3. $a = a^{\perp\perp}$.
4. If $a \leq b^\perp$, then the supremum $a \vee b$ exists in L.
5. If $a \leq b$, then $b = a \vee (b \wedge a^\perp)$ (orthomodular law).

(See, e.g., Pták and Pulmannová, 1991.) Elements $a, b \in L$ are said to be *orthogonal* if $a \leq b^\perp$. A *state* s (finitely additive probability measure) of a *logic* L is defined as a mapping s: $L \to [0, 1]$ such that $s(1) = 1$ and $s(a \vee b) = s(a) + s(b)$, whenever a and b are orthogonal. The logic L is said to be *unital* if for each nonzero element $a \in L$ there is a state s of L such that $s(a) = 1$. On the other hand, a *state* ρ (positive normalized functional) *of an algebra* M is defined as a functional on M such that $\rho(1) = 1$ and $\rho(a^2) \geq 0$ whenever $a \in M$. It is straightforward to show that by restricting a state of M to its projection logic, we get a state of $P(M)$. The central question of the noncommutative measure theory is whether or not all states arise this way. This outstanding problem, known as the Mackey–Gleason problem (Gleason, 1957; Mackey, 1963), is more than thirty years old. A positive answer would establish a relation between a measure and a state analogous to the Riesz representation theorem in classical integration theory. After considerable effort by many authors the Mackey–Gleason question has been answered in the affirmative for all JBW-algebras without type I_2 direct summand (Bunce and Wright, 1984, 1989). Despite this progress the problem remains open for general JB-algebras. Motivated by the Mackey–Gleason problem, we say that an algebra M has the *Gleason property* if every state of $P(M)$ extends to a state of M. It turns out that the Gleason property can be reformulated as an extension property between two special logics and then strengthened in some important cases. We pursue this matter in the following section.

2. UNIVERSAL STATE EXTENSION PROPERTY

We say that a logic L has the *universal state extension property* if every state of L extends to a state of an arbitrary larger unital logic K containing L as a sublogic. It has been proved in Pták (1985) that every Boolean algebra has the universal state extension property. In the case of projection logics the universal state extension property implies the Gleason property of a given operator algebra. Indeed, let M be a JB-algebra such that $P(M)$ has the universal state extension property. By employing the structure theory of JB-algebras, we can embed M into its second dual M^{**}. As M^{**} is a JBW-algebra, it is a subalgebra of a direct sum $C(X, M_3^8) \oplus B(H)_{sa}$ (Hanche-Olsen and Stormer, 1984). Here $C(X, M_3^8)$ denotes an exceptional JBW-algebra consisting of all continuous functions from a compact hyperstonean space X into the algebra M_3^8 of all 3×3 matrices over Cayley numbers. The symbol $B(H)_{sa}$ denotes a self-adjoint part of a C^*-algebra of all bounded operators acting on a Hilbert space H. If necessary we can enlarge H such that dim $H \geq 3$. According to the Gleason theorem (Bunce and Wright, 1989; Gleason, 1957), both algebras $C(X, M_3^8)$ and $B(H)_{sa}$ admit the Gleason property and the same is true of their direct sum. Hence, we can extend every state on $P(M)$ to a state on $P(C(X, M_3^8) \oplus B(H)_{sa})$ and thereby to a linear state of M. Thus, the universal state extension property can be viewed as a stronger form of the Gleason property. The main result of Hamhalter (1994a, b) says that, perhaps surprisingly, the Gleason property already implies the universal state extension property for all projection logics with enough projections.

Theorem 2.1 (Hamhalter, 1994a, b). Let M be a unital JB-algebra such that every maximal associative subalgebra of M is the norm closed linear span of its projections. Then $P(M)$ has the Gleason property if and only if $P(M)$ has the universal state extension property.

Let us remark that in the C^*-algebra version of this result maximal associative subalgebras translate as maximal Abelian subalgebras. The stated result illustrates the power of the Gleason theorem. In fact the Gleason theorem guarantees the possibility of extending a state to arbitrary larger nonlinear structure. In particular, von Neumann algebras enjoy this property.

Corollary 2.2 (Hamhalter, 1994a, b). Every projection logic $P(M)$, where M is a JBW-algebra (or von Neumann algebra) without type I_2 direct summand, has the universal state extension property.

In other words, von Neumann algebra projection logic is well behaved as sublogic; passing to any larger logic we do not lose any information about the original state space. In the physical interpretation quantum logic describes the set of all propositions of a given physical system S, while states of this logic correspond to "physical states" of the system S (Mackey, 1963). Let us consider a larger system S' containing a subsystem S, where S is described by a von Neumann projection lattice. Then, given a state ρ of S, we can always prepare S' such that S is simultaneously in the state ρ.

The extension of states guaranteed in Corollary 2.2 can be specified further as being Hahn–Banach extensions in a certain linear structure connected with a given superlogic. We will need the following notions.

Let $S(L)$ be the state space of a logic L (i.e., the convex set of all states of L). Endowed with the topology of pointwise convergence, the space $S(L)$ becomes a compact Hausdorff space. Let $A^b(L)$ stand for the real Banach space of all bounded affine functions on $S(L)$ with supremum norm. Equipped with the pointwise ordering $f \leq g$ if and only if $f(s) \leq g(s)$ for all $s \in S(L)$, and the unit $u_L(s) = 1$ for all $s \in S(L)$, the structure $(A^b(L), \leq, u_L)$ becomes a complete order unit norm space (Alfsen, 1971; Hanche-Olsen and Stormer, 1984). The logic L can be mapped into $A^b(L)$ by means of the canonical evaluation mapping e_L: $L \to A^b(L)$ defined by $e_L(a)(s) = s(a)$ for all $a \in L$, $s \in S$. For example, if M is a JB-algebra then the space $A^b(P(M))$ can be identified with the second dual M^{**} and $e_{P(M)}$ is induced by the canonical mapping of M into its second dual (see, e.g., Hanche-Olsen and Stormer, 1984).

A *state* of $A^b(L)$ (positive normalized functional) is defined as a mapping ρ: $A^b(L) \to R$ satisfying conditions $\rho(f) \geq 0$ whenever $f \geq 0$ and $\rho(u_L) = 1$. If ρ is a state of $A^b(L)$, then $\rho \circ e_L$ is a state of logic L. For, if $a, b \in L$ are orthogonal, then

$$\rho(e_L(a \vee b)) = \rho(e_L(a) + e_L(b)) = \rho(e_L(a)) + \rho(e_L(b))$$

and obviously, $0 \leq e_L(a) \leq u_L$ implies that $0 \leq \rho(e_L(a)) \leq \rho(u_L) = 1$. Note that not all states arise this way. (As a counterexample we can take, e.g., the projection logic of the algebra of 2×2 complex matrices.) Nevertheless for any JB-algebra we have the following simple observation:

Proposition 2.3. Let M be a JB-algebra. Then M has the Gleason property if and only if every state of $P(M)$ is of the form $f \circ e_{P(M)}$, where f is a state of $A^b(P(M))$.

Proof. Suppose that M has the Gleason property. Then any state ρ of $P(M)$ extends to a linear state $\bar{\rho}$ of M. Let us identify $A^b(P(M))$ with M^{**}. A unit $u_{P(M)}$ is then a unit for $A^b(P(M))$. Denoting by q the canonical embedding of M into M^{**}, we get a state $\rho' = \bar{\rho}(q^{-1}(\cdot))$ defined on a closed subspace $q(M)$ of M^{**}. Using now the Hahn–Banach theorem, we can find a norm-preserving extension f of ρ' to a state of $A^b(P(M))$. Since $\|\rho'\| = \rho'(u_{P(M)}) = 1$ and $\|f\| = \|\rho'\| = 1$, we have that $\|f\| = f(u_{P(M)}) = 1$ and so f is a state (Alfsen, 1971, Prop. II1.3, p. 69). Therefore $\rho = f \circ e_{P(M)}$.

Conversely, for any state f on $A^b(P(M))$ the state $q^{-1} \circ f | q(M)$ is a linear extension of $f \circ e_{P(M)}$ to M. The proof is complete.

Having equivalence between the Gleason property and the universal state extension property, we can further improve Theorem 2.1 and Proposition 2.3 in the following way (Hamhalter, 1994a,b):

Theorem 2.4. Let M be a JB-algebra with the Gleason property and such that every maximal associative subalgebra of M is a closed linear span of its projections. Let K be a unital logic containing $P(M)$ as a sublogic. Then $A^b(P(M))$ can be embedded into $A^b(K)$ and every state ρ of $P(M)$ is of the form

$$f \circ e_K | P(M)$$

where f is a state of $A^b(K)$.

By viewing M as a subspace of $A^b(M)$, we can see that the extensions in Corollary 2.2 can be given by Hahn–Banach extensions of the corresponding states of M. With this in mind, we have obtained the equivalence of the following three basic principles:

1. The universal state extension property (quantum-logic version of the Hahn–Banach theorem).
2. The Gleason property.
3. The Hahn–Banach extension property for order unit spaces.

Let us remark at the end of this section that motivated by Proposition 2.3, we can define the Gleason property for a general quantum logic as follows: the unital logic L is said to have the Gleason property if every state of L is of the form $f \circ e_L$, where f is a state of $A^b(L)$. Then for logics having the Gleason property the universal state extension property is again equivalent to the Hahn–Banach extension property. In order to see it, let us suppose that L has the universal state extension property. Then $A^b(L)$ can be identified with a closed subspace of any space $A^b(K)$, where K is a unital logic containing L (Hamhalter, 1994a, Lemma 3). If, moreover, L has the Gleason property, then any state of L is of the form $f \circ e_L$, where f is a state of $A^b(L)$ and so we can find its extension via the Hahn–Banach theorem. Nevertheless the proof of equivalence of the Gleason property and the state extension property in the case of operator algebras is essentially based on properties of JB-algebras (Hamhalter, 1994a) and probably has no direct analogy in general quantum logic settings.

3. EXTENSIONS OF JAUCH–PIRON STATES

In the previous section we considered the extension properties of general states on projection logics. In this part we focus on extensions within a special category of Jauch–Piron states. A state ρ of a JBW-algebra M is called Jauch–Piron if $\rho(e \vee f) = 0$ whenever $e, f \in P(M)$, with $\rho(e) = \rho(f) = 0$. The Jauch–Piron states play an important role in both operator algebra theory and the foundations of quantum physics and have been therefore studied extensively (Amann, 1989; Hamhalter, 1993; Bunce and Hamhalter, 1994a,b; Jauch, 1968; Jauch and Piron, 1969; Raggio, 1981; Rüttimann, 1977). It follows from the extensive research on the continuity properties of Jauch–Piron states (Bunce and Hamhalter, 1994a,b; Hamhalter, 1993) that Jauch–Piron states do not usually extend to Jauch–Piron states. To exemplify this, let us take an infinite-dimensional Abelian subalgebra A of a von Neumann factor M. We can always find a state ρ on A such that $\rho(e_n) = 0$ for some sequence (e_n) of orthogonal projections with sum 1. Since any Jauch–Piron state on a factor has the kernel closed under countable sums of orthogonal projections (Bunce and Hamhalter, 1994a, Corollary 4.10), ρ has no Jauch–Piron extension over M. Thus, the extension problem for Jauch–Piron states is considerably different from the extension problem for general states and we cannot expect any analogy of the universal state extension property. Nevertheless, it is remarkable that under some circumstances relevant for both mathematics and quantum physics, Jauch–Piron states possess a surprisingly good extension property. Namely, we succeeded in proving the following theorem.

Theorem 3.1 (Bunce and Hamhalter, 1994b). Let M be a JW-algebra not containing type I_2 direct summand and acting on a Hilbert space H. Let W be the von Neumann algebra generated by M. Then every Jauch–Piron state of M extends to a Jauch–Piron state of W.

It is commonly assumed in the foundations of quantum physics that the structure of bounded observables of a given system is described by a JW-algebra. Theorem 3.1 says that in the context of Jauch–Piron states (i.e., states with physical interpretation) we can always take advantage of a more special von Neumann algebra system. Theorem 3.1 also makes it possible to transfer all results on Jauch–Piron states (Bunce and Hamhalter, 1994a) on von Neumann algebras to general JW-algebras.

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